Generalization of Einstein–Lovelock theory to higher order dilaton gravity

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Abstract

A higher order theory of dilaton gravity is constructed as a generalization of the Einstein–Lovelock theory of pure gravity. Its Lagrangian contains terms with higher powers of the Riemann tensor and of the first two derivatives of the dilaton. Nevertheless, the resulting equations of motion are quasi–linear in the second derivatives of the metric and of the dilaton. This property is crucial for the existence of brane solutions in the thin wall limit. At each order in derivatives the contribution to the Lagrangian is unique up to an overall normalization. Relations between symmetries of this theory and the $O(d,d)$ symmetry of the string–inspired models are discussed.

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1 Introduction

The equations of motion in the Einstein theory of gravity in 4 space–time dimensions are the most general divergence–free tensor (rank 2) equations bilinear in the first derivatives and linear in the second derivatives of the metric. They can be obtained from the Hilbert–Einstein action which is linear in the Riemann tensor. In more than 4 space–time dimensions, this theory can be generalized to contain higher powers of the Riemann tensor in the action. The corresponding equations of motion involve higher powers of the first derivatives of the metric and are quasi–linear in the second derivatives (all terms are at most linear in the second derivatives, while multiplied by powers of the first derivatives). It has been shown that the contribution to the action of a given order in the Riemann tensor is unique up to an overall normalization. The quadratic contribution is called the Gauss–Bonnet action or the Lanczos action [1]. It has been generalized to higher orders by Lovelock [2]. The quasi–linearity is a very important feature of the Einstein–Lovelock equations of motion. It guarantees that they can be formulated as a Cauchy problem with some constraints on the initial data [3]. On the other hand, it is crucial for the existence of non–singular domain wall solutions in the thin wall limit. This problem for arbitrary order in derivatives was discussed in [4]. Many aspects of the Einstein–Lovelock gravity were discussed in the literature[1].

Higher derivative corrections to the gravity interactions are present in effective Lagrangians obtained from string theories. The first correction has exactly the form of the Gauss–Bonnet term [9], [10]. The lowest order dilaton interactions were added to the Gauss–Bonnet theory in [11]. However, the α' expansion in string theories predicts higher derivative corrections not only for the gravitational interactions, such corrections appear also for the dilaton. The effective action for the dilaton gravity with terms up to four derivatives was given in [12], [13]. The effective action with six derivatives was presented in [14], but its gravitational part has a form different from that of the corresponding Einstein–Lovelock action.

The dilaton gravity at the field theory level has been investigated by many authors. Some of them included also certain higher order corrections. Yet in most cases such corrections were considered only for gravitational interactions and not for the dilaton. Some higher derivative corrections for both the dilaton and the gravitational interactions were considered in [15]–[21] (certain Riemann tensor combinations with dilaton dependent coefficients were analyzed in [22]–[25]). The terms predicted by superstrings up to four derivatives have also been considered in [26]–[28].

The purpose of the present work is to find a generalization of the lowest order dilaton gravity theory to an arbitrary order in derivatives. We start with the Einstein–Lovelock higher order gravity and couple it to the dilaton. There are many ways to do this but we are only interested in the theories where dilaton and gravity interactions

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[1] Quasi–linearity of the Einstein–Gauss–Bonnet theory was reviewed in [5]. A discussion of general quasi–linear differential equations can be found in [6]. For a review on brane–world gravity see eg. [7]. For a discussion of the Lovelock gravity in the context of the equivalence of the Palatini and metric formulations see eg. [8].

are as similar to each other as possible. Equations of motion in such a theory are presented in Section 2. We begin with formulating the conditions which should be fulfilled by such equations. Most of them are simple generalizations of the conditions fulfilled by the Einstein–Lovelock equations of motion. One condition is added in order to eliminate at least some of the possible theories in which the dilaton interactions are not related to the gravitational ones. The equations of motion satisfying all those conditions are constructed in Subsection 2.3. It turns out that at each order those equations are unique up to a numerical normalization. Moreover, they can be obtained by the standard Euler–Lagrange procedure from the Lagrangian presented in Section 3. Section 4 contains the proof that our equations of motion are quasi–linear in the second derivatives of both the metric and the dilaton. The relation between the gravity and the dilaton interactions is discussed in Section 5. We point out that the Lagrangian of our higher order dilaton gravity can be obtained in a simple way from the pure gravity Einstein–Lovelock Lagrangian. We also discuss the relation of the resulting theory to the $O(d,d)$ symmetric theories. We conclude in Section 6. The Appendix contains the explicit formulae for the Lagrangian and the equations of motion up to terms of the sixth order in derivatives.

2 Equations of motion

2.1 Notation

Let us start with introducing certain generalizations of the Kronecker delta and the trace operator which will be used later to make the formulae more compact. The generalized Kronecker delta is defined by

$$
\delta^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_n} = \det \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_1}_{i_2} & \cdots & \delta^{j_1}_{i_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \delta^{j_n}_{i_1} & \delta^{j_n}_{i_2} & \cdots & \delta^{j_n}_{i_n} \end{vmatrix} , \tag{1}
$$

and should be only employed when the spacetime dimensionality D is sufficient: $D \geq n$. Using this definition it is easy to prove some relations among Kronecker deltas of different order. For example:

$$
\delta^{\nu\, j_1 j_2 \ldots j_n}_{\mu\, i_1 i_2 \ldots i_n} = \delta^{\nu}_{\mu} \delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} - \delta^{\nu}_{i_1} \delta^{j_1 j_2 \ldots j_n}_{\mu\, i_2 \ldots i_n} - \delta^{\nu}_{i_2} \delta^{j_1 j_2 \ldots j_n}_{i_1 \mu\, \ldots i_n} - \ldots - \delta^{\nu}_{i_n} \delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots \mu} . \tag{2}
$$

The generalized Kronecker delta can be used to define the following trace–like linear mapping from tenors of rank (n,n) into numbers

$$
\mathcal{T}(M) = \delta^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_n} M^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} , \tag{3}
$$

which reduces to the ordinary trace for $n = 1$. We will also employ an extension of this operation which maps tensors of rank (n,n) into tensors of rank $(1,1)$:

$$
\overline{\mathcal{T}}^{\nu}_{\mu}(M) = \delta^{\nu\, j_1 j_2 \cdots j_n}_{\mu\, i_1 i_2 \cdots i_n} M^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} . \tag{4}
$$

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In the following we will often use $\mathcal{T}$ and $\overline{\mathcal{T}}$ evaluated for products of tensors. In order to clearly distinguish between tensors and their contracted counterparts, we will use $*$ indices to indicate the rank of a tensor. For example, $\mathcal{R}^{**}_{**}$ denotes the rank $(2,2)$ Riemann tensor, and $\Box^*_*\phi$ denotes the rank $(1,1)$ second derivative of the dilaton, while $\mathcal{R}$ is the Ricci scalar and $\Box\phi$ the D'Alembertian acting on the dilaton. Thus, for example,

$$\mathcal{T}\left((\mathcal{R}^{**}_{**})^2\,(\Box^*_*\phi)^2\right) = \delta^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6}_{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}\,\mathcal{R}^{\rho_1\rho_2}_{\sigma_1\sigma_2}\,\mathcal{R}^{\rho_3\rho_3}_{\sigma_3\sigma_4}\,\Box^{\rho_5}_{\sigma_5}\phi\,\Box^{\rho_6}_{\sigma_6}\phi\,, \tag{5}$$

where we used the notation $\mathcal{R}^{\rho_1\rho_2}_{\sigma_1\sigma_2} = \mathcal{R}^{\rho_1\rho_2}{}_{\sigma_1\sigma_2}$ and $\Box^{\rho}_{\sigma}\phi = \nabla^{\rho}\partial_{\sigma}\phi$ to make the formula more compact. It is easy to see that the sequence of tensors appearing in the product argument of $\mathcal{T}$ is not important. Changing such an order is equivalent to interchanging the appropriate columns of indices in the generalized Kronecker delta. On the other hand, interchanging two such columns of indices is equivalent to interchanging the corresponding 2 rows and 2 columns in the determinant in Definition (1). Each interchange of two columns (or two rows) changes the sign of the determinant, hence an even number of interchanges leaves the determinant unchanged.

2.2   Conditions

Now we want to construct the n–th order dilaton gravity equations of motion. They are to be of the form

$$T^{(n)}_{\mu\nu} = 0\,,$$
$$W^{(n)} = 0\,, \tag{6}$$

where the tensor $T^{(n)}_{\mu\nu}$ and the scalar $W^{(n)}$ satisfy the following conditions

  (i) They are combinations of terms with exactly $2n$ derivatives acting on the metric tensor $g_{\mu\nu}$ and on the dilaton field ϕ. There are no derivatives higher than second acting on one object;

 (ii) Tensor $T^{(n)}_{\mu\nu}$ is symmetric in its indices;

(iii) The covariant derivative of the tensor is proportional to the scalar:
      $\nabla_{\nu}T^{\nu(n)}_{\mu} = \mathrm{const}\cdot(\partial_{\mu}\phi)W^{(n)}$ (the energy–momentum tensor is covariantly conserved if the dilaton equation of motion is fulfilled).

It is clear that the above conditions are not sufficient to determine something which could be regarded as an extension of the higher order gravity theory to the dilaton gravity case. For example, all the above conditions are fulfilled by the Einstein–Gauss–Bonnet gravity with only the lowest order terms for the dilaton. We are interested in a theory where the dilaton and the metric are treated in a more symmetric way. It is not obvious how such a symmetry should be defined, because it ought to relate a scalar to a second rank tensor. Or, more precisely, it is supposed to relate the first and second derivatives of the scalar field to the Riemann tensor and its contractions. A simple observation concerning the gravity part is that it contains even–rank tensors

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only. On the other hand, the first derivative of a scalar is a rank–1 tensor. Hence one can expect that in a gravity–dilaton symmetric theory, the first derivative of the dilaton appears only as a 0–rank tensor: $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. However, the feature mentioned above is not invariant under change of variables. Thus, we should specify in which frame it is fulfilled. The theory which relates dilaton to gravity is the string theory so the string frame seems to be a natural choice. Hence our last condition reads:

(iv) In the string–like frame, in which the pure gravity term is multiplied by $\exp(-\phi)$, the first order derivatives of the dilaton appear in the combination $(\partial_\mu\phi)(\partial^\mu\phi)$ only.

The relation of this condition to the $O(d,d)$ symmetry present in many string–inspired theories will be discussed in Section 5.

2.3 Construction

We start our construction with a term in $T_\mu^{\nu\,(n)}$ where all $2n$ derivatives act on the metric tensors. The only pure gravity tensor satisfying Conditions (i)–(iii) (with $W^{(n)}=0$) is, up to normalization, equal to the n-th order Lovelock tensor [2]. Because of Condition (iv), it is most natural to work in the frame in which the gravity term is multiplied by $\exp(-\phi)$. Consequently, the tensor $T_\mu^{\nu\,(n)}$ starts with

$$T_\mu^{\nu\,(n)} = -2^{-(n+1)}e^{-\phi}\delta^{\nu\,\sigma_1...\sigma_{2n}}_{\mu\,\rho_1...\rho_{2n}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-1}\rho_{2n}}_{\sigma_{2n-1}\sigma_{2n}} + \ldots \tag{7}$$

The reason for such a normalization will be explained in the next section. Calculating the divergence of (7), we get

$$\nabla_\nu T_\mu^{\nu\,(n)} = 2^{-(n+1)}e^{-\phi}(\partial_\nu\phi)\delta^{\nu\,\sigma_1...\sigma_{2n}}_{\mu\,\rho_1...\rho_{2n}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-1}\rho_{2n}}_{\sigma_{2n-1}\sigma_{2n}} + \ldots \tag{8}$$

The above term is produced when the derivative acts on $e^{-\phi}$ (derivatives of the Riemann tensor do not contribute due to the Bianchi identity).

The first term in $T_\mu^{\nu\,(n)}$ shown explicitly in (7) can not be the only one. The reason is that the r.h.s. of (8) is not a product of $\partial_\mu\phi$ and a scalar, as Condition (iii) requires. Using Eq. (2) we can rewrite the r.h.s. of (8) as a combination of $(2n+1)$ terms. The one containing the first term from the r.h.s. of (2) is of the desired form but the remaining $2n$ terms have different structures of the index contractions. It turns out that similar terms are also present in the following covariant derivative

$$\nabla_\nu\left[e^{-\phi}\delta^{\nu\,\sigma_1...\sigma_{2n-1}}_{\mu\,\rho_1...\rho_{2n-1}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-3}\rho_{2n-2}}_{\sigma_{2n-3}\sigma_{2n-2}}\Box^{\rho_{2n-1}}_{\sigma_{2n-1}}\phi\right] =$$

$$= -e^{-\phi}(\partial_\nu\phi)\delta^{\nu\,\sigma_1...\sigma_{2n-1}}_{\mu\,\rho_1...\rho_{2n-1}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-3}\rho_{2n-2}}_{\sigma_{2n-3}\sigma_{2n-2}}\Box^{\rho_{2n-1}}_{\sigma_{2n-1}}\phi +$$

$$+ e^{-\phi}\delta^{\nu\,\sigma_1...\sigma_{2n-1}}_{\mu\,\rho_1...\rho_{2n-1}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-3}\rho_{2n-2}}_{\sigma_{2n-3}\sigma_{2n-2}}\left(\nabla_\nu\nabla^{\rho_{2n-1}}\partial_{\sigma_{2n-1}}\phi\right). \tag{9}$$

The second term on the r.h.s. may be rewritten as

$$e^{-\phi}\delta^{\nu\,\sigma_1...\sigma_{2n-1}}_{\mu\,\rho_1...\rho_{2n-1}}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-3}\rho_{2n-2}}_{\sigma_{2n-3}\sigma_{2n-2}}\left(-\tfrac{1}{2}\delta^{\sigma_{2n}}_{\rho_{2n}}R^{\rho_{2n-1}\rho_{2n}}_{\sigma_{2n-1}\nu}\partial_{\sigma_{2n}}\phi\right) =$$

$$= -\tfrac{1}{2}e^{-\phi}(\partial_\nu\phi)\delta^\nu_{\rho_{2n}}\delta^{\sigma_1...\sigma_{2n-1}\sigma_{2n}}_{\rho_1...\rho_{2n-1}\mu}R^{\rho_1\rho_2}_{\sigma_1\sigma_2}\cdots R^{\rho_{2n-3}\rho_{2n-2}}_{\sigma_{2n-3}\sigma_{2n-2}}R^{\rho_{2n-1}\rho_{2n}}_{\sigma_{2n-1}\sigma_{2n}}, \tag{10}$$

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where in the last step we interchanged the names of the contracted indices ν and σ_{2n} and rearranged the indices in the generalized Kronecker delta. A term exactly of this structure must be added to (8) in order to obtain an expression proportional to $\partial_\mu \phi$. From Eqs. (2) and (8) it follows that the coefficient should be equal to $(-n2^{-n})$ instead of the $(-1/2)$ present in (10). This fixes the coefficient of the term in $T_\mu^{\nu\,(n)}$ which contains $(n-1)$ Riemann tensors and one second derivative of the dilaton. Now we know the first two terms of the tensor $T_\mu^{\nu\,(n)}$. Using the notation introduced in (3) and (4), they can be written as:

$$T_\mu^{\nu\,(n)} = -2^{-(n+1)}e^{-\phi}\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^n\right) - 2^{-(n-1)}ne^{-\phi}\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-1)}\,\Box_*^*\phi\right) + \ldots \qquad (11)$$

Their covariant derivative reads

$$\nabla_\nu T_\mu^{\nu\,(n)} = 2^{-(n+1)}e^{-\phi}(\partial_\mu\phi)\overline{\mathcal{T}}\left((\mathcal{R}_{**}^{**})^n\right)$$
$$+2^{-(n-1)}ne^{-\phi}(\partial_\nu\phi)\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-1)}\,\Box_*^*\phi\right) + \ldots \qquad (12)$$

The first term has the structure required by Condition (iii) and determines the first term of the scalar equation of motion[2] $W^{(n)}$. However, the second term in (12) is not of the appropriate structure. It means that some additional terms, whose covariant derivatives are products of $(n-1)$ Riemann tensors with one second derivative of the dilaton, are necessary in $T_\mu^{\nu\,(n)}$. Two such terms are possible:

$$c_3 e^{-\phi}\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-2)}\,(\Box_*^*\phi)^2\right) + c_4 e^{-\phi}\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-1)}\right)(\partial\phi)^2\,. \qquad (13)$$

However, it is not enough to have terms with appropriate powers of the Riemann tensor and the dilaton, because their covariant divergences must contain the correct combinations of the generalized Kronecker deltas. To check whether this is possible, we calculate the covariant divergence of (13). When the derivative acts on $\Box_*^*\phi$ in the first term in (13), it gives an additional Riemann tensor multiplied by $\partial\phi$ and a pair of new indices. Those new indices are contracted with just one ordinary Kronecker delta and are not under the overall antisymmetrization. Similarly, when the covariant derivative acts on $(\partial\phi)^2$ in the second term in (13), it gives the second derivative of the dilaton multiplied by $\partial\phi$ and a pair of new indices. Those two covariant derivatives should combine with the second term on the r.h.s. of (12) to give an expression proportional to $\partial_\mu\phi$. This fixes the numerical coefficients c_3 and c_4. The explicit calculation gives $c_3 = -2^{(2-n)}n(n-1)$, $c_4 = 2^{-n}n$. Thus, we have found the first four terms of $T_\mu^{\nu\,(n)}$:

$$T_\mu^{\nu\,(n)} = -2^{-(n+1)}e^{-\phi}\left[\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^n\right) + 4n\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-1)}\,\Box_*^*\phi\right)+\right.$$

$$+8n(n-1)\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-2)}\,(\Box_*^*\phi)^2\right)+$$

$$\left.-2n\overline{\mathcal{T}}_\mu^{\,\nu}\left((\mathcal{R}_{**}^{**})^{(n-1)}\right)(\partial\phi)^2\right] + \ldots \qquad (14)$$

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[2] up to an overall normalization. The choice of the relative normalizations of $T_{\mu\nu}^{(n)}$ and $W^{(n)}$ shall become clear when the Lagrangian is introduced in Section 3.

The covariant divergence of those terms reads

$$\nabla_\nu T_\mu^{\nu\,(n)} = \partial_\mu\phi \left\{ 2^{-(n+1)}e^{-\phi} \left[\mathcal{T}\left((\mathcal{R}_{**}^{**})^n\right) + 4n\mathcal{T}\left((\mathcal{R}_{**}^{**})^{(n-1)}\,\Box_*^*\phi\right) \right] \right\}$$

$$+ \partial_\nu\phi\, 2^{(2-n)}n(n-1)e^{-\phi}\overline{\mathcal{T}}_\mu^\nu\left((\mathcal{R}_{**}^{**})^{(n-2)}\,(\Box_*^*\phi)^2\right)$$

$$- \partial_\nu\phi\, 2^{-n}ne^{-\phi}\overline{\mathcal{T}}_\mu^\nu\left((\mathcal{R}_{**}^{**})^{(n-1)}\right)(\partial\phi)^2 + \dots \tag{15}$$

The terms in the curly bracket above are the first two terms of the scalar $W^{(n)}$ we are looking for.

Equation (15) shows that the procedure of finding $T_\mu^{\nu(n)}$ and $W^{(n)}$ must be continued. The last two terms on the r.h.s. of (15) do not have the required form, so more terms must be added to $T_\mu^{\nu(n)}$. From the steps described so far, it should be clear that each of such new terms must contain exactly 3 (first or second order) derivatives of the dilaton. There are two such terms:

$$c_5 e^{-\phi}\overline{\mathcal{T}}_\mu^\nu\left((\mathcal{R}_{**}^{**})^{(n-3)}\,(\Box_*^*\phi)^3\right) + c_6 e^{-\phi}\overline{\mathcal{T}}_\mu^\nu\left((\mathcal{R}_{**}^{**})^{(n-2)}\,\Box_*^*\phi\right)(\partial\phi)^2\,. \tag{16}$$

The coefficients c_5 and c_6 can be fixed in the same way as c_3 and c_4.

This procedure can be continued step by step for the terms containing higher and higher powers of the dilaton field with the derivatives acting on it. Eventually, one obtains the term with the maximal number of dilaton fields, namely $c\,e^{-\phi}\delta_\mu^\nu\,[(\partial\phi)^2]^n$. This is the first term in $T_\mu^{\nu(n)}$, the covariant derivative of which need not to be corrected by contributions from any additional terms. This covariant derivative reads

$$\nabla_\nu\left[e^{-\phi}\delta_\mu^\nu\,[(\partial\phi)^2]^n\right] = -(\partial_\mu\phi)e^{-\phi}\left[(\partial\phi)^2\right]^n + 2ne^{-\phi}(\nabla_\mu\partial^\sigma\phi)(\partial_\sigma\phi)\left[(\partial\phi)^2\right]^{(n-1)}\,. \tag{17}$$

The second term on the r.h.s. is used to cancel some unwanted part of $\nabla_\mu\left[e^{-\phi}\overline{\mathcal{T}}_\mu^\nu(\Box_*^*\phi)\,[(\partial\phi)^2]^{(n-1)}\right]$, which fixes c to be equal to $\frac{1}{2}(-1)^{(n+1)}$. The first term on the r.h.s. of (17) has already the required structure of the product of $\partial_\mu\phi$ and a scalar. Thus, the procedure can stop here.

The above iterative procedure gives $T_{\mu\nu}^{(n)}$ and $W^{(n)}$ satisfying all the four imposed conditions. The resulting gravitational and dilaton equations of motion can be written in the following relatively simple form:

$$T_{\mu\nu}^{(n)} = -\frac{1}{2}e^{-\phi}\sum_{a=0}^n\sum_{b=0}^{n-a}\frac{2^{b-a}n!}{a!b!(n-a-b)!}\overline{\mathcal{T}}_{\mu\nu}\left((\mathcal{R}_{**}^{**})^a\,(\Box_*^*\phi)^b\right)\left(-(\partial\phi)^2\right)^{n-a-b} =$$

$$= 0\,, \tag{18}$$

$$W^{(n)} = -e^{-\phi}\sum_{a=0}^n\sum_{b=0}^{n-a}\frac{2^{b-a}n!}{a!b!(n-a-b)!}\mathcal{T}\left((\mathcal{R}_{**}^{**})^a\,(\Box_*^*\phi)^b\right)\left(-(\partial\phi)^2\right)^{n-a-b} =$$

$$= 0\,. \tag{19}$$

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The existence of $T^{(n)}_{\mu\nu}$ and $W^{(n)}$ is a non–trivial result, because in our iterative procedure there are more conditions than available constants. A priori it could happen that there were no solutions other than a trivial one with vanishing $T^{(n)}_{\mu\nu}$ and $W^{(n)}$. However, the solution exists and is unique up to an overall normalization. Hence any dilaton gravity equations of motion, satisfying Conditions (i)–(iv), which contain at least one term present in (18) and (19) must also contain all the other terms with uniquely determined coefficients.

3 Lagrangian

It is interesting to check whether the equations of motion constructed in Section 2 can be obtained from some D–dimensional action. In such case, $T^{(n)}_{\mu\nu}$ and $W^{(n)}$ would satisfy

$$\delta_{g^{\mu\nu}} S^{(n)} = \delta_{g^{\mu\nu}} \int \mathrm{d}^D x \sqrt{-g}\, \mathcal{L}^{(n)} = \int \mathrm{d}^D x \sqrt{-g}\, T^{(n)}_{\mu\nu} \delta g^{\mu\nu}, \qquad (20)$$

$$\delta_\phi S^{(n)} = \delta_\phi \int \mathrm{d}^D x \sqrt{-g}\, \mathcal{L}^{(n)} = \int \mathrm{d}^D x \sqrt{-g}\, W^{(n)} \delta \phi. \qquad (21)$$

It turns out that indeed the equations of motion (18) and (19) can be obtained from the action with the Lagrangian density given by

$$\mathcal{L}^{(n)} = e^{-\phi} \sum_{a=0}^{n} \sum_{b=0}^{n-a} \frac{2^{b-a} n!}{a! b! (n-a-b)!}\, \mathcal{T}\left((\mathcal{R}^{**}_{**})^a \left(\Box^*_* \phi \right)^b \right) \left(-(\partial\phi)^2 \right)^{n-a-b}. \qquad (22)$$

It is important to underline that for Conditions (i)–(iv) not to be violated, the terms coming from the n-th Lagrangian can appear only in the space–times with dimensionality $D \geq 2n$. Moreover, one should be careful when calculating (20) for $D = 2n$, as the generalized Kronecker delta (1) can not be employed in (18) for the term of the highest order in the Riemann tensor. The coefficient of that term should be replaced with

$$\delta^{\nu\,\sigma_1\sigma_2...\sigma_{2n}}_{\mu\,\rho_1\rho_2...\rho_{2n}} \xrightarrow[D=2n]{} \delta^\nu_\mu \delta^{\sigma_1\sigma_2...\sigma_{2n}}_{\rho_1\rho_2...\rho_{2n}} - \delta^\nu_{\rho_1} \delta^{\sigma_1\sigma_2...\sigma_{2n}}_{\mu\,\rho_2...\rho_{2n}} - \delta^\nu_{\rho_2} \delta^{\sigma_1\sigma_2...\sigma_{2n}}_{\rho_1\mu\,...\rho_{2n}} - \ldots - \delta^\nu_{\rho_{2n}} \delta^{\sigma_1\sigma_2...\sigma_{2n}}_{\rho_1\rho_2...\mu}. \qquad (23)$$

Now we can comment on the overall normalization of the tensors $T^{(n)}_{\mu\nu}$. The reason for this particular normalization is that the term $e^{-\phi} \mathcal{R}^n$ (with $\mathcal{R}$ being the Ricci scalar) appears in the Lagrangian with the coefficient 1. This corresponds to the standard normalizations of the Hilbert–Einstein and Gauss–Bonnet Lagrangians.

Proving that the equations of motion derived from the Lagrangian (22) really have the form (6) with $T^{(n)}_{\mu\nu}$ and $W^{(n)}$ as given in (18) and (19) is a straightforward but quite tedious calculation. One of the reasons is that apparently several integrations by parts are required. This can be somewhat simplified if one observes that not all those integrations by parts have to be performed explicitly. In case of (21), the reason is as follows. Under the integral (21) there are first (second) derivatives of $\delta\phi$ coming from the variation of the first (second) derivatives of the dilaton. In general, the terms

containing second derivatives of $\delta\phi$ should be integrated by parts twice. However, one can notice that the result of a single integration and the terms containing the first derivatives of $\delta\phi$ cancel each other exactly.

The situation is a little bit more complicated in case of the gravitational equation of motion. Under the integral (20), there are second derivatives of $\delta g^{\mu\nu}$ coming from the variation of the Riemann tensor and first derivatives of $\delta g^{\mu\nu}$ coming from the variation of the second covariant derivative of the dilaton. Similarly as in the case of the dilaton equation of motion, the terms containing second derivatives of $\delta g^{\mu\nu}$ have to be integrated by parts only once. And although the cancellation of the resulting terms is not complete this time, only some residual integration by parts has to be performed additionally.

Of course, the Lagrangian density (22) is not unique. First, one can rewrite $\mathcal{L}^{(n)}$ changing the variables $g_{\mu\nu}$ and ϕ. Second, one can add to $\mathcal{L}^{(n)}$ any total divergence without changing the resulting equations of motion. However, the form given in Eq. (22) is especially simple and interesting. It is very similar to the form of $T_{\mu\nu}^{(n)}$ and $W^{(n)}$. The energy momentum tensor $T_{\mu\nu}^{(n)}$ can be obtained from $\mathcal{L}^{(n)}$ by replacing the generalized trace $\mathcal{T}$ with its tensor extension $\overline{\mathcal{T}}_{\mu\nu}$ and multiplying the result by $-1/2$. In case of the dilaton equation of motion, the analogous relation is even simpler: $W^{(n)} = -\mathcal{L}^{(n)}$.

We were not able to find any other similarly simple form of the Lagrangian by adding total derivative terms or by changing the variables. For example, we examined the form of the Lagrangian and of the equations of motion in the Einstein–like frame in which the common factor $e^{-\phi}$ is absorbed by a suitable Weyl transformation. The results are very complicated and will not be presented here. One of the reasons for such complications is that the Weyl transformation depends on the dimensionality D of the space–time. Thus, many different functions of D appear in the Einstein frame, while there is no explicit dependence on D in our string–like frame.

4 Quasi-linearity

It is easy to show that the equations of motion (18)–(19) are quasi–linear in the second derivatives of the metric and the dilaton. Let us introduce in the D–dimensional space–time a $(D-1)$–dimensional hypersurface Σ defined by its unit normal vector n^μ. The metric induced at this hypersurface is given by

$$h_{\mu\nu} = g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}, \tag{24}$$

where $n^2 = n_\rho n^\rho$. The components of the D–dimensional Riemann tensor $\mathcal{R}_{**}^{**}$ corresponding to the full metric $g_{\mu\nu}$ can be expressed as

$$\mathcal{R}_{\mu\nu}^{\rho\sigma} = R_{\mu\nu}^{\rho\sigma} - n^{-2}\left(2K_{[\mu}^{[\rho}K_{\nu]}^{\sigma]} + 4n^{[\rho}D_{[\mu}K_{\nu]}^{\sigma]} + 4n_{[\mu}D^{[\rho}K_{\nu]}^{\sigma]}\right)$$

$$+ n^{-4}\left(4n_{[\mu}n^{[\rho}K_{|\tau|}^{\sigma]}K_{\nu]}^{\tau} - 4n_{[\mu}n^{[\rho}\pounds_n K_{\nu]}^{\sigma]}\right), \tag{25}$$

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where: R is the $(D-1)$–dimensional Riemann tensor corresponding to the induced metric $h_{\mu\nu}$; K is the extrinsic curvature given by

$$K_{\mu\nu} = \frac{1}{2}\pounds_n h_{\mu\nu} \; ; \tag{26}$$

D_μ is the covariant derivative with respect to the induced metric $h_{\mu\nu}$; $\pounds_n$ is the Lie derivative along the vector field n^μ. Similarly we can write the D–dimensional second covariant (with respect to the metric $g_{\mu\nu}$) derivative of the dilaton

$$\nabla_\mu\nabla_\nu\phi = D_\mu D_\nu\phi + n^{-2}\left(K_{\mu\nu}\pounds_n\phi + 2n_{(\mu}D_{\nu)}\pounds_n\phi - 2n_{(\mu}K_{\nu)}^\tau D_\tau\phi\right)$$
$$+ n^{-4}\,n_\mu n_\nu\left(\pounds_n^2\phi - (n^\rho\nabla_\rho n^\tau)\nabla_\tau\phi\right). \tag{27}$$

We want to check how the second Lie derivatives of the metric $h_{\mu\nu}$ (present in $\pounds_n K_{\mu\nu}$) and of the dilaton ϕ appear in the equations of motion (18) and (19). Such second derivatives are present in (25) and (27) but in both cases they are multiplied by coefficients bilinear in the vector n. After substituting the decompositions (25) and (27) into (18) and (19), one can immediately see that, due to the antisymmetrization present in $T_{\mu\nu}^{(n)}$ and $W^{(n)}$, the equations of motion contain terms at most bilinear in n. Thus, the equations of motion (18) and (19) contain terms at most linear in the second Lie derivatives of $h_{\mu\nu}$ and ϕ.

We have shown that the equations of motion are quasi–linear in the second Lie derivatives "perpendicular" to the hypersurface Σ. This quasi–linearity has very important consequences. For Σ with a time–like n, this allows us to define a standard Cauchy problem with the initial conditions (values and first Lie derivatives of $h_{\mu\nu}$ and ϕ) given at Σ. For a space–like n, the quasi–linearity is necessary to have non–singular brane solutions even in the thin wall limit.

5   Symmetries

The equations of motion presented in Section 2 were obtained assuming some kind of symmetry between the metric and the dilaton. Now we are in a position to investigate such a symmetry in more detail.

It is quite amazing that the Lagrangian (22) as well as the equations of motion (18) and (19) can be expressed as functions of n–th perfect "power" of one simple n–independent quantity. Namely:

$$\mathcal{L}^{(n)} = -W^{(n)} = e^{-\phi}\mathcal{T}\left[\left(\frac{1}{2}\mathcal{R}_{**}^{**} \oplus 2\Box_*^*\phi \oplus (-1)\left(\partial\phi\right)^2\right)^n\right], \tag{28}$$

$$T_{\mu\nu}^{(n)} = -\frac{1}{2}e^{-\phi}\overline{\mathcal{T}}_{\mu\nu}\left[\left(\frac{1}{2}\mathcal{R}_{**}^{**} \oplus 2\Box_*^*\phi \oplus (-1)\left(\partial\phi\right)^2\right)^n\right]. \tag{29}$$

One can treat these equations as just a new notation allowing us to rewrite the double

sums from (18), (19) and (22) in a compact way[3]. Yet, on the other hand, it helps to show that the action and the equations of motion depend on some combinations of the dilaton derivatives and tensors obtained from the metric only. In each round parenthesis in Eqs. (28) and (29), there are the rank–4 Riemann tensor, the rank–2 tensor of the second derivatives of the dilaton and the rank–0 tensor built from the first derivatives of the dilaton:

$$\frac{1}{2}\mathcal{R}^{**}_{**} \oplus 2\Box^*_*\phi \oplus (-1)(\partial\phi)^2 . \tag{30}$$

All those tensors are under the generalized traces $\mathcal{T}$ and $\overline{\mathcal{T}}_{\mu\nu}$. Some of the terms present in these mappings contain traces of the tensors from (30). There are two different rank–2 tensors coming from (30). The first is just $\Box^*_*\phi$. The second is the Ricci tensor $\mathcal{R}^*_*$ which can be obtained from the Riemann tensor by contraction of its two indices. There are four different ways to contract one pair of indices in $\mathcal{R}^{**}_{**}$, thus in the final result the rank–2 tensors appear always in the combination $(2\mathcal{R}^*_* + 2\Box^*_*\phi)$. There are three different scalars originating from (30): $(\partial\phi)^2$, $\Box\phi$ and the curvature scalar $\mathcal{R}$. There are two different constructions giving $\mathcal{R}$, so the final results depend on a single following scalar combination: $\mathcal{R} + 2\Box\phi - (\partial\phi)^2$.

The above observation allows us to relate our dilaton gravity equations to the corresponding equations in the pure Einstein–Lovelock gravity:

$$\mathcal{L}^{(n)} = -W^{(n)} = e^{-\phi}\mathcal{L}^{(n)}_{E-L}\left[\mathcal{R}^{**}_{**}, (\mathcal{R}^*_* + \Box^*_*\phi), (\mathcal{R} + 2\Box\phi - (\partial\phi)^2)\right], \tag{31}$$

$$T^{(n)}_{\mu\nu} = e^{-\phi}\left(T^{(n)}_{E-L}\right)_{\mu\nu}\left[g_{\mu\nu}, \mathcal{R}^{**}_{**}, (\mathcal{R}^*_* + \Box^*_*\phi), (\mathcal{R} + 2\Box\phi - (\partial\phi)^2)\right]. \tag{32}$$

The recipe for the higher order dilaton gravity can be as follows: Start with the higher order pure gravity Einstein–Lovelock theory. Write the Lagrangian density $\mathcal{L}^{(n)}_{E-L}$ and the equations of motion $(T^{(n)}_{E-L})_{\mu\nu}$ in terms of the Riemann tensor, Ricci tensor and the curvature scalar by performing all internal (within a given Riemann tensor) contractions of indices. Then make the substitutions:

$$\mathcal{R}^\sigma_\rho \rightarrow \left[\mathcal{R}^\sigma_\rho + \Box^\sigma_\rho\phi\right] , \tag{33}$$

$$\mathcal{R} \rightarrow \left[\mathcal{R} + 2\Box\phi - (\partial\phi)^2\right] . \tag{34}$$

Finally, multiply the result by $\exp(-\phi)$. The dilaton equation of motion, absent in the pure gravity case, is simply $\mathcal{L}^{(n)} = 0$.

It occurs that the form of the Lagrangian and the tensor $T^{(n)}_{\mu\nu}$ given in (31) and (32) is very closely related to the string $O(d,d)$ symmetry[4]. To show this, we consider the

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[3] One could say that Eqs. (28) and (29) make no sense because they contain a sum of tensors of different ranks. To make this mathematically sensible, we should consider a simple sum of spaces of tensors of a given rank. Then the tensors in (28) and (29) should be understood as elements of such a sum space with all but one components set to zero. Finally, the generalized traces $\mathcal{T}$ and $\overline{\mathcal{T}}_{\mu\nu}$ should be further extended in such a way that when acting on an element of this big space they give the result being the sum of generalized traces of all components.

[4] For a review on $O(d,d)$ symmetry, see e.g. [29] and the references therein.

D–dimensional block–diagonal metric of the form

$$g_{\mu\nu} = \begin{pmatrix} \tilde{g}_{\alpha\beta} & 0 \\ 0 & G_{mn} \end{pmatrix}, \tag{35}$$

where $\alpha, \beta = 1, \ldots, (D - d)$; $m, n = (D - d + 1), \ldots, D$. We assume that the metric components $\tilde{g}_{\alpha\beta}$ and G_{mn} and the dilaton field ϕ do not depend on the last d coordinates x^m. In such a case, we obtain the following expressions for the second derivatives of the dilaton

$$\Box_\alpha^\beta \phi = \tilde{\Box}_\alpha^\beta \phi, \tag{36}$$

$$\Box_m^n \phi = \frac{1}{2} \left(G^{-1} \partial_\alpha G \right)_m^n \partial^\alpha \phi, \tag{37}$$

$$\Box \phi = \tilde{\Box} \phi + \frac{1}{2} \left(\partial_\alpha \ln \det G \right) \partial^\alpha \phi, \tag{38}$$

and for the Ricci tensor and the curvature scalar

$$\mathcal{R}_\alpha^\beta = \tilde{\mathcal{R}}_\alpha^\beta - \frac{1}{2} \tilde{\Box}_\alpha^\beta \ln \det G - \frac{1}{4} \text{Tr} \left[G^{-1} \left(\partial_\alpha G \right) G^{-1} \left(\partial^\beta G \right) \right], \tag{39}$$

$$\mathcal{R}_m^n = -\frac{1}{4} \left(\partial_\alpha \ln \det G \right) \left(G^{-1} \partial^\alpha G \right)_m^n - \frac{1}{2} \left(G^{-1} \tilde{\Box} G \right)_m^n$$
$$+ \frac{1}{2} \left[G^{-1} \left(\partial_\alpha G \right) G^{-1} \left(\partial^\alpha G \right) \right]_m^n, \tag{40}$$

$$\mathcal{R} = \tilde{\mathcal{R}} - \frac{1}{4} \left(\partial_\alpha \ln \det G \right) \left(\partial^\alpha \ln \det G \right) - \tilde{\Box} \ln \det G$$
$$- \frac{1}{4} \text{Tr} \left[G^{-1} \left(\partial_\alpha G \right) G^{-1} \left(\partial^\alpha G \right) \right], \tag{41}$$

where tilde denotes quantities related to the $(D - d)$–dimensional metric $\tilde{g}_{\alpha\beta}$, G should be understood as a $d \times d$ matrix (and not its determinant) and Tr and det are the trace and the determinant (acting on $d \times d$ matrices).

A necessary condition for the $O(d, d)$ symmetry is that the dilaton field ϕ appear only in the $O(d, d)$ invariant combination

$$\Phi = \phi - \frac{1}{2} \ln \det G. \tag{42}$$

Hence any derivative of the dilaton ϕ must be accompanied by an appropriate derivative of $[\ln \det G]$. It is easy to see that there are only three combinations of Eqs. (36)–(41) and the first derivatives of ϕ which depend on ϕ and $[\ln \det G]$ only through the combination Φ:

$$\mathcal{R} + 2\Box\phi - (\partial\phi)^2 = \tilde{\mathcal{R}} + 2\tilde{\Box}\Phi - \partial_\alpha\Phi\partial^\alpha\Phi - \frac{1}{4}\text{Tr}\left[G^{-1} \left(\partial_\alpha G \right) G^{-1} \left(\partial^\alpha G \right) \right], \tag{43}$$

$$\mathcal{R}_\alpha^\beta + \Box_\alpha^\beta\phi = \tilde{\mathcal{R}}_\alpha^\beta + \tilde{\Box}_\alpha^\beta\Phi - \frac{1}{4}\text{Tr}\left[G^{-1} \left(\partial_\alpha G \right) G^{-1} \left(\partial^\beta G \right) \right], \tag{44}$$

$$\mathcal{R}_m^n + \Box_m^n\phi = \frac{1}{2}\left(\partial_\alpha\Phi \right)\left(G^{-1}\partial^\alpha G \right)_m^n - \frac{1}{2}\tilde{\nabla}_\alpha\left(G^{-1}\partial^\alpha G \right)_m^n. \tag{45}$$

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These are exactly the combinations which, together with the Riemann tensor with uncontracted indices, appear in the formulation given in Eqs. (31) and (32). Hence the higher derivative contributions to the dilaton gravity theory found in the present paper fulfill the necessary condition for the $O(d, d)$ symmetry formulated before Eq. (42). This does not mean yet that our theory is a part of some $O(d, d)$ symmetric theory. One should check whether all terms depending on G other than $[\ln \det G]$ form only $O(d, d)$ invariant combinations. Actually, one can calculate that it is really the case for $n = 1$ and $n = 2$. The lowest order theory was analyzed from this point of view for the first time in [30]. Our second order Lagrangian $\mathcal{L}^{(2)}$ differs from the one found in [31] (for a vanishing tensor field H) by some total derivatives only. Thus, for $n = 1, 2$ the equations of motion presented in Section 2 are the same as the dilaton and gravity part of the equations obtained as appropriate approximations from the superstring theories. The relation to the $O(d, d)$ symmetry for $n > 2$ will be discussed elsewhere [32].

The above discussion shows that Condition (iv) from Section 2.2 can be treated as a necessary one for the dilaton gravity model to be part of some $O(d, d)$ symmetric theory. The reason is that there are no $O(d, d)$ invariant expressions containing the first derivatives of the dilaton other than the combination $(\partial_\mu \phi)(\partial^\mu \phi)$.

6   Conclusions

We have generalized the Einstein–Lovelock theory by adding interactions with the dilaton. The corresponding Einstein and dilaton equations of motion can be written as series in the number of derivatives acting on the fields:

$$T_{\mu\nu} = \sum_n \kappa_n T_{\mu\nu}^{(n)} = 0 \,, \tag{46}$$

$$W = \sum_n \kappa_n W^{(n)} = 0 \,. \tag{47}$$

The n–th contributions $T_{\mu\nu}^{(n)}$ and $W^{(n)}$ are sums of terms containing products of the Riemann tensor and the first and second derivatives of the dilaton field. There are $2n$ derivatives in each such term. We have found the most general equations of motion satisfying Conditions (i)–(iv) given in Section 2.2. The first three conditions are the standard properties of the dilaton gravity theories. The last one was added in order to find the theories in which the dilaton and the metric are treated, as much as possible, on the same footing. Accordingly, we assumed that the rank–1 tensor containing the first derivatives of the dilaton can appear only in the scalar combination $(\partial_\mu \phi)(\partial^\mu \phi)$, as there is no way to build an odd–rank tensor from the metric and the Riemann tensor. It is necessary to specify the frame in which such a condition is to be fulfilled. We have chosen the string frame where the n–th order term from the Einstein–Lovelock theory is multiplied by $e^{-\phi}$. The reason is quite simple: symmetries relating the dilaton and the metric are present in string–motivated theories.

We have shown that at each order $T_{\mu\nu}^{(n)}$ and $W^{(n)}$ are unique up to a normalization. General expressions for $T_{\mu\nu}^{(n)}$ and $W^{(n)}$ for arbitrary n are given in Section 2.3. The explicite formulae for $n \leq 3$ are presented in the Appendix. It occurs that the higher order dilaton gravity equations of motion have properties similar to those of the pure Einstein–Lovelock gravity. Namely:

- There is an upper limit on the number of terms in (46)–(47) which can be non–zero. For a D–dimensional space–time it is given by the inequality $2n \leq D$ (the corresponding limit for pure gravity is $2n < D$)

- The equations of motion are quasi–linear in the second derivatives. This allows us to treat them as a standard Cauchy initial conditions problem. It is crucial also for the existence of brane–type solutions in the thin wall limit.

There is also another very interesting feature of those equations. The form of the scalar and Einstein equations is very similar when written with the help of the generalized Kronecker delta. The tensor $T_{\mu\nu}^{(n)}$ can be obtained from the scalar $W^{(n)}$ simply by adding a pair of extra indices μ and ν to each generalized Kronecker delta and dividing by 2.

Our dilaton gravity equations of motion can be obtained from an appropriate Lagrangian. Of course, such a Lagrangian can be determined only up to some total derivatives. However, we have found that there is one particularly interesting form of it:

$$\mathcal{L} = -W \, . \tag{48}$$

Moreover, this Lagrangian is related in a simple way to the Einstein–Lovelock one (the same is true also for the gravitational equations of motion). First, one has to write the Einstein–Lovelock Lagrangian as a function of the Riemann tensor, the Ricci tensor and the curvature scalar by performing all internal (within the same Riemann tensor) contractions of indices. Next, one should replace the curvature scalar with the combination $\mathcal{R} + 2\Box\phi - (\partial\phi)^2$, and the Ricci tensor with $\mathcal{R}_\rho^\sigma + \Box_\rho^\sigma$. The result is the dilaton gravity Lagrangian.

The property that the Lagrangian can be written in terms of only three tensors: one scalar $\mathcal{R} + 2\Box\phi - (\partial\phi)^2$, one rank–2 tensor $\mathcal{R}_\rho^\sigma + \Box_\rho^\sigma$ and the rank–4 Riemann tensor is quite important. We have shown that this is a necessary condition for the dilaton gravity to be a part of any string motivated theory with the $O(d, d)$ symmetry. It turns out that for $n = 1, 2$ it is also a sufficient one. The contributions $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ to our Lagrangian are, up to total derivatives, the same as those found from demanding the $O(d, d)$ symmetry [30], [31]. It would be interesting to investigate the relation of $\mathcal{L}^{(n)}$ to string theories for $n > 2$ [32].

Most of the interesting features of the Lagrangian and the equations of motion are visible in the string frame only. The theory looks more complicated in other frames. For example, in the most often used Einstein frame there are no simple relations between tensors built from the metric and from the dilaton derivatives and also many coefficients become explicitly D–dependent. The advantages of the string frame should not be surprising. For example, much more explicit solutions in the lowest order

dilaton gravity were found in the string frame [33] than in the Einstein one (discussions concerning the relation between the string and the Einstein frames are reviewed in [34]). Our results show that the string frame is the most convenient one to investigate dilaton gravity also at higher orders.

Acknowledgments

Appendix

The dilaton gravity Lagrangian and the corresponding equations of motion can be written as a series in the number of derivatives

$$\mathcal{L} = \sum_{n=0}^{[D/2]} \kappa_n \mathcal{L}^{(n)} , \tag{A.1}$$

$$T_\mu^\nu = \sum_{n=0}^{[D/2]} \kappa_n T_\mu^{\nu(n)} = 0 , \tag{A.2}$$

and the dilaton equation of motion $W = -\mathcal{L} = 0$.
The 0–th order terms correspond to the cosmological constant:

$$e^\phi \mathcal{L}^{(0)} = 1 \ , \tag{A.3}$$

$$e^\phi T_\mu^{\nu(0)} = -\frac{1}{2}\delta_\mu^\nu . \tag{A.4}$$

The 1–st order contribution is the standard Einstein gravity interacting with the dilaton:

$$e^\phi \mathcal{L}^{(1)} = \mathcal{R} + 2\Box\phi - (\partial\phi)^2 , \tag{A.5}$$

$$e^\phi T_\mu^{\nu(1)} = -\frac{1}{2}\delta_\mu^\nu e^\phi \mathcal{L}^{(1)} + \left(\mathcal{R}_\mu^\nu + \Box_\mu^\nu \phi\right). \tag{A.6}$$

The next two orders are given by the following expressions:

$$e^{\phi}\mathcal{L}^{(2)} = \left(e^{\phi}\mathcal{L}^{(1)}\right)^2 - 4\left(\mathcal{R}^{\rho_2}_{\rho_1} + \Box^{\rho_2}_{\rho_1}\phi\right)\left(\mathcal{R}^{\rho_1}_{\rho_2} + \Box^{\rho_1}_{\rho_2}\phi\right) + \mathcal{R}^{\rho_2\rho_4}_{\rho_1\rho_3}\mathcal{R}^{\rho_1\rho_3}_{\rho_2\rho_4}, \tag{A.7}$$

$$e^{\phi}T^{\nu(2)}_{\mu} = -\frac{1}{2}\delta^{\nu}_{\mu}e^{\phi}\mathcal{L}^{(2)} + 2\left(\mathcal{R}^{\nu}_{\mu} + \Box^{\nu}_{\mu}\phi\right)e^{\phi}\mathcal{L}^{(1)} - 4\left(\mathcal{R}^{\rho}_{\mu} + \Box^{\rho}_{\mu}\phi\right)\left(\mathcal{R}^{\nu}_{\rho} + \Box^{\nu}_{\rho}\phi\right)$$
$$-4\mathcal{R}^{\nu\rho_2}_{\mu\rho_1}\left(\mathcal{R}^{\rho_1}_{\rho_2} + \Box^{\rho_1}_{\rho_2}\phi\right) + 2\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_2}_{\rho_1\rho_3}, \tag{A.8}$$

$$e^{\phi}\mathcal{L}^{(3)} = 3\left(e^{\phi}\mathcal{L}^{(2)}\right)\left(e^{\phi}\mathcal{L}^{(1)}\right) - 2\left(e^{\phi}\mathcal{L}^{(1)}\right)^3$$
$$+16\left(\mathcal{R}^{\rho_2}_{\rho_1} + \Box^{\rho_2}_{\rho_1}\phi\right)\left(\mathcal{R}^{\rho_3}_{\rho_2} + \Box^{\rho_3}_{\rho_2}\phi\right)\left(\mathcal{R}^{\rho_1}_{\rho_3} + \Box^{\rho_1}_{\rho_3}\phi\right)$$
$$+24\left(\mathcal{R}^{\rho_2}_{\rho_1} + \Box^{\rho_2}_{\rho_1}\phi\right)\left(\mathcal{R}^{\rho_4}_{\rho_3} + \Box^{\rho_4}_{\rho_3}\phi\right)\mathcal{R}^{\rho_1\rho_3}_{\rho_2\rho_4} - 24\left(\mathcal{R}^{\rho_2}_{\rho_1} + \Box^{\rho_2}_{\rho_1}\phi\right)\mathcal{R}^{\rho_1\rho_5}_{\rho_3\rho_4}\mathcal{R}^{\rho_3\rho_4}_{\rho_2\rho_5}$$
$$-8\mathcal{R}^{\rho_2\rho_4}_{\rho_1\rho_3}\mathcal{R}^{\rho_1\rho_6}_{\rho_2\rho_5}\mathcal{R}^{\rho_3\rho_5}_{\rho_4\rho_6} + 2\mathcal{R}^{\rho_2\rho_4}_{\rho_1\rho_3}\mathcal{R}^{\rho_1\rho_3}_{\rho_5\rho_6}\mathcal{R}^{\rho_5\rho_6}_{\rho_2\rho_4}, \tag{A.9}$$

$$e^{\phi}T^{\nu(3)}_{\mu} = -\frac{1}{2}\delta^{\nu}_{\mu}e^{\phi}\mathcal{L}^{(3)} + 3\left(\mathcal{R}^{\nu}_{\mu} + \Box^{\nu}_{\mu}\phi\right)e^{\phi}\mathcal{L}^{(2)} - 12\mathcal{R}\left(\mathcal{R}^{\rho}_{\mu} + \Box^{\rho}_{\mu}\phi\right)\left(\mathcal{R}^{\nu}_{\rho} + \Box^{\nu}_{\rho}\phi\right)$$
$$-12\mathcal{R}\mathcal{R}^{\nu\rho_2}_{\mu\,\rho_1}\left(\mathcal{R}^{\rho_1}_{\rho_2} + \Box^{\rho_1}_{\rho_2}\phi\right) + 6\mathcal{R}\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_2}_{\rho_1\rho_3}$$
$$+24\left(\mathcal{R}^{\rho_1}_{\mu} + \Box^{\rho_1}_{\mu}\phi\right)\left(\mathcal{R}^{\nu}_{\rho_2} + \Box^{\nu}_{\rho_2}\phi\right)\left(\mathcal{R}^{\rho_2}_{\rho_1} + \Box^{\rho_2}_{\rho_1}\phi\right)$$
$$+24\mathcal{R}^{\nu\rho_2}_{\mu\,\rho_1}\left(\mathcal{R}^{\rho_3}_{\rho_1} + \Box^{\rho_3}_{\rho_1}\phi\right)\left(\mathcal{R}^{\rho_2}_{\rho_3} + \Box^{\rho_2}_{\rho_3}\phi\right) + 24\mathcal{R}^{\nu\rho_2}_{\mu\,\rho_1}\mathcal{R}^{\rho_1\rho_4}_{\rho_2\rho_3}\left(\mathcal{R}^{\rho_3}_{\rho_4} + \Box^{\rho_3}_{\rho_4}\phi\right)$$
$$+24\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\left(\mathcal{R}^{\nu}_{\rho_1} + \Box^{\nu}_{\rho_1}\phi\right)\left(\mathcal{R}^{\rho_2}_{\rho_3} + \Box^{\rho_2}_{\rho_3}\phi\right) - 12\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\rho_4\rho_2}_{\rho_1\rho_3}\left(\mathcal{R}^{\nu}_{\rho_4} + \Box^{\nu}_{\rho_4}\phi\right)$$
$$+24\mathcal{R}^{\nu\,\rho_3}_{\rho_1\rho_2}\left(\mathcal{R}^{\rho_1}_{\mu} + \Box^{\rho_1}_{\mu}\phi\right)\left(\mathcal{R}^{\rho_2}_{\rho_3} + \Box^{\rho_2}_{\rho_3}\phi\right) - 12\mathcal{R}^{\nu\,\rho_3}_{\rho_1\rho_2}\mathcal{R}^{\rho_1\rho_2}_{\rho_4\rho_3}\left(\mathcal{R}^{\rho_4}_{\mu} + \Box^{\rho_4}_{\mu}\phi\right)$$
$$-24\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_2}_{\rho_1\rho_4}\left(\mathcal{R}^{\rho_4}_{\rho_3} + \Box^{\rho_4}_{\rho_3}\phi\right) - 12\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_3}_{\rho_4\rho_2}\left(\mathcal{R}^{\rho_4}_{\rho_3} + \Box^{\rho_4}_{\rho_3}\phi\right)$$
$$-12\mathcal{R}^{\nu\rho_2}_{\mu\,\rho_1}\mathcal{R}^{\rho_1\rho_5}_{\rho_3\rho_4}\mathcal{R}^{\rho_3\rho_4}_{\rho_2\rho_5} + 6\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_2}_{\rho_4\rho_5}\mathcal{R}^{\rho_4\rho_5}_{\rho_1\rho_3} - 24\mathcal{R}^{\rho_1\rho_3}_{\mu\,\rho_2}\mathcal{R}^{\nu\,\rho_5}_{\rho_4\rho_3}\mathcal{R}^{\rho_4\rho_2}_{\rho_1\rho_5}. \tag{A.10}$$

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